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Inhomogeneous Inverse Differential Realization of Multimode $SU(2)$ Group

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The generators and irreducible representation coherent state of the multimode $SU(2)$ group are constructed by using the inverse operators of the multimode bosonic harmonic oscillator, and the inhomogeneous inverse differential realization of the multimode $SU(2)$ group are derived.

It is well known that the boson realization approach is very effective for studying the representation theory of groups, and the boson realization usually can be obtained from the creation and annihilation operators of the bosonic harmonic oscillator, such as the Jordan–Schwinger realization, etc. On the other hand, the nature of the inverse operator of the bosonic harmonic oscillator (Dirac, 1966) has been studied, and some new results have been given in recent articles (Mehta and Roy, 1992; Fan, 1993, 1994). In consideration of the close relationship between the quantum mechanical quasi-accuration problem and the inhomogeneous differential realization of the Lie group (Turbiner and Ushveridge, 1987; Turbiner, 1988), the present paper studies the inhomogeneous inverse differential realization (a new kind of inhomogeneous differential realization) of the multimode $SU(2)$ group by using the inverse operator of the multimode bosonic harmonic oscillator.

We first introduce a k-mode bosonic operator as follows:

$$A_k = a_1 a_2 \cdots a_k \left\{ \frac{n_1^a n_2^a \cdots n_k^a}{\min(n_1^a, n_2^a, \ldots, n_k^a)} \right\}^{-1/2} \tag{1}$$

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where

$$a_i^+ a_i = n_i^a, \qquad a_i a_i^+ = n_i^a + 1, \qquad [a_i, a_i^+] = 1 \tag{2}$$

$$[n_i^a, a_i] = -a_i, \qquad [n_i^a, a_i^+] = a_i^+ \tag{3}$$

It is easy to check

$$[A_k, A_k^+] = 1 \tag{4}$$

$$[N_k^a, A_k^+] = A_k^+, \qquad [N_k^a, A_k] = -A_k \tag{5}$$

where $N_k^a = \min(n_1^a, n_2^a, \ldots, n_k^a)$. Therefore (4) and (5) denote that $\{A_k^+, A_k, N_k^a\}$ can be regarded as a k-mode bosonic harmonic oscillator.

Similar to Fan's works (Fan, 1993, 1994), the inverses of the k-mode bosonic harmonic oscillator creation and annihilation operators can be obtained from their actions on the number states $|n, n, \ldots\rangle$ (where $|n, n, \ldots\rangle = |n\rangle_1 |n\rangle_2 \ldots |n\rangle_k$):

$$A_k^{-1}|n, n, \ldots\rangle = \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle \tag{6}$$

$$(A_k^+)^{-1}|n, n, \ldots\rangle = \begin{cases} \dfrac{1}{\sqrt{n}} |n-1, n-1, \ldots\rangle & (n \neq 0) \\ 0 & (n = 0) \end{cases} \tag{7}$$

which means

$$A_k^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n+1, n+1, \ldots\rangle\langle n, n, \ldots| \tag{8}$$

$$(A_k^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n, n, \ldots\rangle\langle n+1, n+1, \ldots| = (A_k^{-1})^+ \tag{9}$$

A noncommutative relation between A_k and A_k^{-1} follows:

$$A_k A_k^{-1} = (A_k^+)^{-1} A_k^+ = 1 \tag{10}$$

$$A_k^{-1} A_k = A_k^+ (A_k^+)^{-1} = 1 - |0, 0, \ldots\rangle\langle 0, 0, \ldots| \tag{11}$$

which means that A_k^{-1} is the right inverse of A_k and $(A^+)_k^{-1}$ is the left inverse of A_k^+.

We now introduce another k-mode bosonic harmonic oscillator $\{B_k^+, B_k, N_k^b\}$ as follows:

$$B_k = b_1 b_2 \cdots b_k \left\{ \frac{n_1^b n_2^b \cdots n_k^b}{\min(n_1^b, n_2^b, \ldots, n_k^b)} \right\}^{-1/2} \tag{12}$$

where

$$b_i^+ b_i = n_i^b, \qquad b_i b_i^+ = n_i^b + 1, \qquad [b_i, b_i^+] = 1 \tag{13}$$

$$[n_i^b, b_i] = -b_i, \qquad [n_i^b, b_i^+] = b_i^+ \tag{14}$$

it is easy to find that

$$[B_k, B_k^+] = 1 \tag{15}$$

$$[N_k^b, B_k^+] = B_k^+, \qquad [N_k^b, B_k] = -B_k \tag{16}$$

The nature of the inverses for B_k and B_k^+ are the same as with A_k and A_k^+.

Consider the combinations of the inverses between $\{A_k^+, A_k, N_k^a\}$ and $\{B_k^+, B_k, N_k^b\}$, namely

$$J_+^{-1} = (A_k^+)^{-1}B_k^{-1}, \qquad J_-^{-1} = (B_k^+)^{-1}A_k^{-1} \tag{17}$$

$$J_0^{-1} = \tfrac{1}{2}[(N_k^a + 1)^{-1}(N_k^b)^{-1} - (N_k^a)^{-1}(N_k^b + 1)^{-1}] \tag{18}$$

where

$$(N_k^a)^{-1} = A_k^{-1}(A_k^+)^{-1}, \qquad (N_k^a + 1)^{-1} = (A_k^+)^{-1}A_k^{-1} \tag{19}$$

$$(N_k^b)^{-1} = B_k^{-1}(B_k^+)^{-1}, \qquad (N_k^b + 1)^{-1} = (B_k^+)^{-1}B_k^{-1} \tag{20}$$

which satisfy the relations

$$(N_k^a)^{-1}|n, n, \ldots\rangle = \frac{1}{n}|n, n, \ldots\rangle, \qquad (N_k^a + 1)^{-1}|n, n, \ldots\rangle = \frac{1}{n+1}|n, n, \ldots\rangle \tag{21}$$

$$(N_k^b)^{-1}|n, n, \ldots\rangle = \frac{1}{n}|n, n, \ldots\rangle; \qquad (N_k^b + 1)^{-1}|n, n, \ldots\rangle = \frac{1}{n+1}|n, n, \ldots\rangle \tag{22}$$

Therefore (17) and (18) give the commutation relations

$$[J_0^{-1}, J_\pm^{-1}] = \pm J_\pm^{-1}, \qquad [J_+^{-1}, J_-^{-1}] = 2J_0^{-1} \tag{23}$$

This indicates that the operators J_+^{-1}, J_-^{-1}, and J_0^{-1} construct a closed multimode SU (2) group; one can say that they are the Jordan–Schwinger realizations.

The unitary irreducible representations $|j, m; j, m; \ldots\rangle$ of the multimode $SU(2)$ group are

$$|j, m; j, m; \ldots\rangle = |j + m, j + m, \ldots\rangle \otimes |j - m, j - m, \ldots\rangle \qquad (-j \leq m \leq j) \tag{24}$$

These irreducible representations are finite and depend on a single quantum number $j = 0, 1/2, 1, \ldots$ The actions of the multi-mode $SU(2)$ group generators on the elements of the irreducible representation (24) are given by

$$J_+^{-1}|j, m; j, m; \ldots\rangle = \frac{1}{\sqrt{(j+m)(j-m+1)}} \times |j, m-1; j, m-1; \ldots\rangle \tag{25}$$

$$J_-^{-1}|j, m; j, m; \ldots\rangle = \frac{1}{\sqrt{(j+m+1)(j-m)}} \times |j, m+1; j, m+1; \ldots\rangle \tag{26}$$

$$J_0^{-1}\,|j, m; j, m; \ldots\rangle = \frac{m}{(j+m+1)(j+m)(j-m)(j-m+1)} \times |j, m; j, m; \ldots\rangle \tag{27}$$

The coherent state of the irreducible representation for the multimode $SU(2)$ group is written as

$$|jZ\rangle = e^{2J_-^{-1}}|j, -j; j, -j; \ldots\rangle = \sum_{m=-j}^{j} \frac{1}{(j+m)!}\sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, Z^{j+m}|j, m; j, m; \ldots\rangle \tag{28}$$

It is easy to obtain the normalization coefficient $C_j(|z|^2)$ for the coherent state

$$C_j(|z|^2) = \sum_{m=-j}^{j} \frac{(j-m)!}{(2j)![(j+m)!]^3}(|z|^2)^{j+m} \tag{29}$$

The method we used here is based on an idea of Yu *et al.* (1995). In order to construct the completeness relation of the quantum state $|jz\rangle$, we define $P(j+m, z)$ to be an observable probability of $|j,m; j,m; \ldots\rangle$ in state $|jz\rangle$, i.e.,

$$P(j+m, z) = |\langle j, m; j, m; \cdots |jz\rangle|^2 = \frac{(j-m)!}{(2j)![(j+m)!]^3}(|z|^2)^{j+m} \tag{30}$$

Setting $P(j+m) = \int P(j+m, z)\, dz^2$ and letting ρ represent the density matrix of state $|j, m; j, m; \ldots\rangle$, we have

$$\rho = \sum_{m=-j}^{j} P(j+m)|j, m; j, m; \ldots\rangle\langle j, m; j, m; \ldots| \tag{31}$$

Thus the completeness relation of the quantum state $|jz\rangle$ can be written as

$$\frac{1}{\pi}\rho^{-1}\int\frac{|jz)(jz|}{C_j(|z|^2)}\,dz^2 = 1 \tag{32}$$

We now define the Bargmann–Fock representation of the bases $|j, m; j, m; \ldots\rangle$ for the irreducible representation as follows:

$$\begin{aligned} f_{jm}(z) &= (j\bar{z}|j, m; j, m; \cdots\rangle \\ &= \frac{1}{(j+m)!}\sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, z^{j+m} \end{aligned} \tag{33}$$

Furthermore, we define a state vector in the space of the irreducible representation

$$|\psi\rangle = \sum_{m=-j}^{j} C_m|j, m; j, m; \ldots\rangle \tag{34}$$

We have

$$\begin{aligned} (j\bar{z}|J_-^{-1}|\psi\rangle &= \sum_m C_m(j\bar{z}|J_-^{-1}|j, m; j, m; \ldots\rangle \\ &= \sum_m \frac{C_m}{(j+m+1)^2(j+m)!(j-m)}\sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, z^{j+m+1} \end{aligned} \tag{35}$$

On the other hand, we also have by using the inverse derivative formula (Ye, 1979),

$$\begin{aligned} &\frac{1}{(2j - z\, d/dz)(1/z)(z\, d/dz)^2}(j\bar{z}|\psi\rangle \\ &\quad = \sum_m \frac{C_m}{(j+m+1)^2(j+m)!(j-m)}\sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, z^{j+m+1} \end{aligned} \tag{36}$$

From (35) and (36), we have the inhomogeneous inverse differential realization of the operator J_-^{-1},

$$B_j(J_-^{-1}) = \frac{1}{(2j - z\, d/z)(1/z)(z\, d/dz)^2} \tag{37}$$

Similarly, we also have

$$\begin{aligned} (j\bar{z}|J_+^{-1}|\psi\rangle &= \sum_m C_m(j\bar{z}|\, J_+^{-1}|j, m; j, m; \ldots\rangle \\ &= \sum_m \frac{C_m}{(j+m-1)!}\sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, z^{j+m-1} \end{aligned} \tag{38}$$

$$\frac{d}{dz}\langle j\bar{z}|\psi\rangle = \sum_m \frac{C_m}{(j+m-1)!}\sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, z^{j+m-1} \tag{39}$$

$$\begin{aligned}\langle j\bar{z}|J_0^{-1}|\psi\rangle &= \sum_m C_m \langle j\bar{z}|J_0^{-1}|j, m; j, m; \ldots\rangle \\ &= \sum_m \frac{mC_m}{(j+m)!(j+m)(j+m+1)(j-m)(j-m+1)} \\ &\quad \times \sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, z^{j+m}\end{aligned} \tag{40}$$

$$\begin{aligned}&\left(z\frac{d}{dz}-j\right)\frac{1}{(2j-z\,d/dz)(1/z)(2j-z\,d/dz)(d/dz)(z^2)(d/dz)}\langle j\bar{z}|\Psi\rangle \\ &= \sum_m \frac{mC_m}{(j+m)!(j+m)(j+m+1)(j-m)(j-m+1)} \\ &\quad \times \sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}\, z^{j+m}\end{aligned} \tag{41}$$

Therefore we can obtain the inhomogeneous inverse differential realizations of the operators J_+^{-1} and J_0^{-1}, respectively,

$$B_j(J_+^{-1}) = \frac{d}{dz} \tag{42}$$

$$B_j(J_0^{-1}) = \left(z\frac{d}{dz}-j\right)\frac{1}{(2j-z\,d/dz)(1/z)(2j-z\,d/dz)(d/dz)(z^2)(d/dz)} \tag{43}$$

Finally, we point out that Eqs. (37), (42), and (43) are the inhomogeneous inverse differential realizations of the multimode $SU(2)$ group.

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